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Dispersion Relations for IMPATT Diodes

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ABSTRACT

This report clarifies the nature of the dispersion relation for space-charge waves in IMPATT diodes. It is demonstrated that, in the usual linear approximation, the dispersion relation is always cubic, although suitable transformations of the basic equation appear to yield a quadratic. The implications of this point are discussed in regard to prior results and for a simple, but tractable, generalization of these results.

In addition, possible implications for the TRAPATT-ARP controversy concerning the explanation of anomalous mode operation of avalanche diodes are discussed.

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Franklin C. Hudson
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CONTENTS

DISPERSION RELATIONS FOR IMPATT DIODES

This report is concerned with a clarification of the basic nature of IMPATT wave dispersion relations which apparently are not widely appreciated,[1] along with a generalization of some prior results. In the first section, three alternative sets of basic equations will be derived and discussed.

I. BASIC EQUATIONS

One basic set of equations for one-dimensional interactions is the continuity equations for electron and hole currents along with the Poisson equation. These are[2]

$$\frac{\partial n}{\partial t} = \frac{1}{q}\frac{\partial J_n}{\partial x} + \alpha n|V_n| + \beta p|V_p| \quad , \tag{1}$$

$$\frac{\partial p}{\partial t} = \frac{-1}{q}\frac{\partial J_p}{\partial x} + \alpha n|V_n| + \beta p|V_p| \quad , \tag{2}$$

$$\frac{\partial E_x}{\partial x} = \frac{q}{\epsilon}(N_o - N_a) + \frac{q}{\epsilon}(p - n) \quad , \tag{3}$$

where $J_n = -qn|V_n|$ and $J_p = -qp|V_p|$ are the electron and hole current densities, n and p are the electron and hole densities, q is the electronic charge, $|V_n|$ and $|V_p|$ are the magnitudes of the electron and hole velocities, α and β are the electron and hole ionization rates, N_o and N_a are the time-invariant donor and acceptor densities as determined by the material doping, ϵ is the permittivity, and E_x is the electric field. This set involves three equations and three unknowns (n, p, E_x), and hence the dispersion relation can be expected to be cubic when the equations are linearized.

A second set can be derived in at least two ways. In the first way, Eq. (2) is subtracted from Eq. (1) to yield

$$\frac{\partial(n-p)}{\partial t} = \frac{1}{q}\frac{1}{\partial x}(J_n + J_p) \quad . \tag{4}$$

If Eq. (3) is differentiated partially with respect to time, the result is

$$\frac{\partial^2 E_x}{\partial x\,\partial t} = \frac{q}{\epsilon}\frac{\partial}{\partial t}(p - n) \quad . \tag{5}$$

Substituting Eq. (5) into Eq. (4), we obtain

$$\epsilon\frac{\partial^2 E_x}{\partial x\,\partial t} + \frac{\partial}{\partial x}(J_n + J_p) = 0 \quad , \tag{6}$$

which may be partially integrated with respect to x to yield

$$J_n + J_p + \epsilon\frac{\partial E_x}{\partial t} = J_T \quad , \tag{7}$$

where J_T, called the total current density, represents the sum of the particle and displacement current densities and is solely a function of time. Equations (1), (2), and (7) constitute the alternative set of three equations in three unknowns (n, p, E_x) for which a cubic dispersion relation is expected when the equations are linearized. The second method of derivation recalls that the Maxwell equations,

$$\bar{\nabla} \times \bar{H} = \bar{J}_n + \bar{J}_p + \epsilon \frac{\partial \bar{E}_x}{\partial t} \quad , \tag{8}$$

$$\nabla \times \bar{E} = -\mu_o \frac{\partial \bar{H}}{\partial t} \quad , \tag{9}$$

must be applicable to E_x in Eq. (3). Taking the divergence of Eq. (8), we obtain

$$\bar{\nabla} \cdot \left\{ \bar{J}_n + \bar{J}_p + \epsilon \frac{\partial \bar{E}_x}{\partial t} \right\} = 0 \tag{10}$$

because $\bar{\nabla} \cdot \bar{\nabla} \times \bar{H} = 0$ is a vector identity. For one-dimensional interactions ($\bar{\nabla} = \hat{x}_o \frac{\partial}{\partial x}$, $\bar{E} = E_x \hat{x}_o$, $\bar{J}_n = J_n \hat{x}_o$, $\bar{J}_p = J_p \hat{x}_o$), Eq. (10) becomes Eq. (6) and hence Eq. (7).

In the small-signal time-harmonic case where

$$E_x = E_o + E_1 e^{j\omega t} \quad , \tag{11}$$

$$J_n = J_{no} + J_{n1} e^{j\omega t} \quad , \tag{12}$$

$$J_p = J_{po} + J_{p1} e^{j\omega t} \quad , \tag{13}$$

$$\alpha = \alpha_o + \alpha'_o E_1 e^{j\omega t} \quad , \tag{14}$$

$$\alpha_o \equiv \alpha(Eo) \quad , \quad \alpha'_o \equiv [d\alpha/dE]_{E=E_o} \quad ,$$

the equations for the time-harmonic components are

$$j\omega n_1 = -|V_n| \frac{\partial n_1}{\partial x} + g_1 \quad , \tag{15}$$

$$j\omega p_1 = |V_p| \frac{\partial p_1}{\partial x} + g_1 \quad , \tag{16}$$

$$\frac{\partial E_1}{\partial x} = \frac{q}{\epsilon} (p_1 - n_1) \quad , \tag{17}$$

(where $g_1 \equiv \alpha_o n_1 |V_n| + \beta_o p_1 |V_p| + \alpha'_o E_1 N_o |V_n| + \beta'_o E_1 P_o |V_p|$) or Eqs. (15), (16) and

$$J_{T1} = J_{n1} + J_{p1} + j\omega\epsilon E_1 \quad , \tag{18}$$

(where $\partial J_{T1}/\partial x = 0$). It is important to note that Eq. (17) can be derived from Eqs. (15), (16), and (18). This could not be done in the large-signal case because Eqs. (1), (2), and (6) combine to give

$$\frac{\partial^2 E_x}{\partial x \, \partial t} = \frac{q}{\epsilon} \frac{\partial (p-n)}{\partial t} \quad ,$$

which integrates to

$$\frac{\partial E_x}{\partial x} = \frac{q}{\epsilon} (p - n) + F(x) \quad ,$$

where F(x) is an unknown function [unknown because information not contained in Eqs. (1), (2), and (6) must be invoked to prove that $F(x) = (q/\epsilon)(N_o - N_a)$].

Some investigators regard the derivation of Eq. (17) from Eqs. (15), (16), and (18) as indication that there are only two independent equations [Eqs. (15) and (16)] and hence that the dispersion relation must be quadratic.[1] They apparently neglect to include Eq. (18) in their equation count. It is the form of the third alternative set of equations, which we now derive, which apparently encourages the preceding mistaken belief.

If Eq. (17) is differentiated with respect to x to obtain

$$\frac{\partial^2 E_1}{\partial x^2} = \frac{q}{\epsilon}\left(\frac{\partial p_1}{\partial x} - \frac{\partial n_1}{\partial x}\right) \quad , \tag{19}$$

and Eqs. (15) and (16) are substituted in Eq. (19), the result is

$$\frac{\partial^2 E_1}{\partial x^2} + \frac{2q}{\epsilon}\left(\alpha'_o N_o + \beta'_o P_o\right) E_1 = \frac{q}{\epsilon}\left\{\frac{j\omega}{|V_p|} - \beta_o\left[1 + \left|\frac{V_p}{V_n}\right|\right]\right\} P_1$$

$$+ \frac{q}{\epsilon}\left\{\frac{j\omega}{|V_n|} - \alpha_o\left[1 + \left|\frac{V_n}{V_p}\right|\right]\right\} n_1 \quad . \tag{20}$$

In the very special case where

$$\alpha_o = \beta_o \quad ,$$

$$|V_n| = |V_p| = V_s \quad ,$$

then

$$\frac{\partial^2 E_1}{\partial x^2} + AE_1 = BJ_{T1} \quad , \tag{21}$$

where

$$A = \left(\frac{2}{\epsilon V_s}\right)\left(\alpha'_o J_{no} + \beta'_o J_{po} - \frac{\omega^2 \epsilon}{V_s} - 2j\omega\epsilon\alpha_o\right) \quad ,$$

$$B = \left(\frac{2}{\epsilon V_s}\right)\left(\frac{j\omega}{V_s} - 2\alpha_o\right) \quad ,$$

are constants if E_o and hence α_o, β_o, α'_o and β'_o are independent of x as in the case of a PIN configuration. Because Eq. (21) is a second-order, inhomogeneous differential equation for E, some investigators[1] take this as further evidence that the dispersion relation must be quadratic. This view overlooks the point that the solution to Eq. (21) will include a constant term (i.e., independent of x) due to the presence of the constant "forcing function," BJ_{T1}, in the differential equation. This constant term in the solution corresponds to the K = 0 root (when substituted into e^{-jKx}) of the cubic dispersion relation. If the cubic dispersion relation did not possess a K = 0 root, the reduction of the system of three equations [Eqs. (15), (16), and (17)] to the single second-order differential equations, Eq. (21), would not be possible. Thus, the solutions to the three coupled first-order equations have the form

$$\{n_1, p_1, E_1\} = \sum_{K_i=K_1}^{K_3} B_i e^{-jK_i x} = A_1 e^{-jK_1 x} + A_2 e^{-jK_2 x} + A_3 e^{-jK_3 x} \quad , \tag{22}$$

which, if one uses the information that $K_1 = 0$, becomes

$$\{n_1, p_1, E_1\} = A_1 + A_2 e^{-jK_2 x} + A_3 e^{-jK_3 x} \quad , \tag{23}$$

which is the form of the solution for Eq. (21). The basic third-order, homogeneous differential equation is derived in the Appendix.

II. DISPERSION RELATION

Manasse and Shapiro[3] have explored, at length, a generalization of Misawa's dispersion relation[2] which is more physically realistic, while still remaining tractable, because it takes into account the difference in electron and hole parameters. This dispersion relation is[3]

$$K^2 \left\{ K^2 d + K\left[\frac{\omega}{V}(1-d) + j\alpha_o d(C-1)\right] + \left[\frac{\alpha'_o J_o (1+d)}{\epsilon V_s} - (\omega/V_s)^2 - j\omega\alpha_o(1+cd)/V_s\right]\right\} = 0 \quad , \tag{24}$$

where it has been assumed that $J_o \equiv qv(n_o + cdP_o) = \text{constant}$, and

$$\frac{\beta'_o}{\alpha'_o} = \frac{\beta_o}{\alpha_o} \equiv c \quad , \quad \frac{|V_n|}{|V_p|} \equiv d \quad , \quad |V_n| \equiv V_s \quad . \tag{25}$$

However, Manasse and Shapiro ignore the double root at $K = 0$, of Eq. (24), and do not regard it as part of the true solution. Rather they, as well as others, considered it to be quadratic, apparently for some of the reasons discussed in the first section.[1]

We consider now, briefly, a simple and tractable generalization of the preceding results. We note, from Sze and Gibbons,[4] that α'_o/α_o need not equal β'_o/β_o, and in fact differ by a factor of approximately two for silicon. Thus, from Eqs. (15), (16), and (17), we find for traveling-wave solutions

$$e^{j(\omega t - Kx)}$$

that

$$K\left\{K^2 d + K\left[\frac{\omega}{V_s}(d-1) + jd(\alpha_o - \beta_o d)\right] + \left[-(\omega/V_s)^2 + (\alpha'_o J_{no} + \beta'_o J_{po})(1+d)/\epsilon V_s - j\left(\frac{\omega}{V_s}\right)(\beta_o d + \alpha_o)\right]\right\} = 0 \quad , \tag{26}$$

where $V_s = |V_n| = |V_p|/d$. When $\alpha_o/\beta_o = \alpha'_o/\beta'_o$, Eq. (26) reduces to Eq. (24) except for an extra factor of d in the third term, ... + $jd(\alpha_o + \beta_o d)]$, of the left-hand side of Eq. (26) whose absence in Eq. (24) is presumably due to a typographical error. The solutions to Eq. (26) are

$$K = 0 \quad , \tag{27}$$

and

$$K = \frac{\omega}{2V_s d}(d-1) + j\frac{(\alpha_o - \beta_o d)}{2} \pm \frac{1}{2d}\left\{\left[\frac{\omega}{V_s}(d-1) + j(\alpha_o - \beta_o d)\, d\right]^2 + 4d\left[\frac{(\alpha'_o J_{no} + \beta'_o J_{po})(1+d)}{\epsilon V_s} - (\omega/V_s)^2 - j(\omega/V_s)(\beta_o d + \alpha_o)\right]\right\}^{1/2} \quad . \tag{28}$$

The inverse dispersion relation, $\omega = g(K)$ is obtained from

$$\omega^2 + \omega[j(\beta_o d + \alpha_o) + K(d-1)] V_s + \left[\frac{(1+d)}{\epsilon V_s} (\alpha_o' J_{no} + \beta_o' J_{po}) + jKd(\alpha_o - \beta_o d) + K^2 d\right] V_s^2 = 0 \quad . \tag{29}$$

A description of $\alpha_o' J_{no} + \beta_o' J_{po}$, given J_{dc}, is required to make use of the above. This requires a straightforward generalization of the argument given in Appendix II of Ref. 3. The result is from an averaging procedure for the dc results and yields for $\alpha_o' J_n + \beta_o' J_{po} \equiv G$,

$$G \cong \alpha_o' J_{dc} [1 + (\frac{e-1}{1-c}) (1 - 1/\alpha_o L)] \quad , \tag{30}$$

where $c \equiv \beta_o / \alpha_o$, $e \equiv \beta_o' / \alpha_o'$, and L is the length of the avalanche region. When $e = c$, the result reduces to Manasse and Shapiro's $J_o = J_{dc}/\alpha_o L$.

The preceding result was obtained by using $\alpha_o' J_{no} + \beta_{po} = \alpha_o' [J_{dc} + (e-1) J_{po}]$ (recall that $J_{dc} = J_{no} + J_{po}$), and[3]

$$J_{po} = \left[\frac{e^{(\alpha_o - \beta_o)L} - e^{(\alpha_o - \beta_o)x}}{e^{(\alpha_o - \beta_o)L} - 1}\right] \quad , \tag{31}$$

$$G = \frac{1}{L} \int_0^L (\alpha_o' J_{no} + \beta_o' J_{po}) \, dx \quad , \tag{32}$$

$$\beta_o / \alpha_o = e^{-(\alpha_o - \beta_o)L} \quad \text{(at breakdown)} \quad . \tag{33}$$

III. CONCLUSIONS

We consider now the application of some of the preceding results to a controversial topic. It has been demonstrated that IMPATT diode operation is described by a third-order homogeneous, differential equation which gives rise to a cubic dispersion relation. This conclusion is contrary to the widely held belief, by workers in the field,[1] that the basic equation and resulting dispersion relation are of second order. This distinction does not appear to be of practical importance for small-signal theory of IMPATT diodes, since all three roots of the dispersion relation $(0, +k_m, -k_m)$ are used in practice by everyone to calculate diode properties.

There may be important consequences in large-signal avalanche diode theory. At the present time, proponents of the TRAPATT mode theory[5] (as an explanation of anomalous mode avalanche diode behavior) argue that they have solved the general system of equations by computer simulation and see only TRAPATT mode operation. Other workers[6] claim that there exists an additional high-efficiency mode, avalanche-resonance pumped (ARP), which they obtain experimentally and which does not have the waveforms (of J_o and voltage vs time) predicted by TRAPATT theory.

The solution to the controversy may lie in the observation that the computer simulations of TRAPATT mode operation assume given waveforms for J_T (typically a step-modulated sine wave) which is equivalent to solving the system as though it were described completely by a second-order partial-differential equation. It seems plausible to this author that if J_T were not specified, but treated as the unknown it truly is, a more general solution might be obtained which would contain ARP and TRAPATT modes as separate possible solutions. It may be argued in opposition to this view that it is known that J_T, in the one-dimensional case, is solely a function of time and that the assumed time dependence for J_T can be arrived at by reasonable physical arguments. This is true but not conclusive, since it is pointed out on page 2 that in the large-signal case Eqs. (1), (2), and (6) are not equivalent to (in the sense that they cannot be used to derive) Eqs. (1), (2), and (3). Thus, assumptions on the form of $J_T(t)$ may omit other physically possible but not obvious solutions.

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APPENDIX
DERIVATION OF BASIC EQUATION

The basic set of three linearized, first-order, coupled differential equations, Eqs. (15), (16), and (17), combine to yield a third-order, homogeneous differential equation which we display in this appendix.

Eqs. (15) and (16) may be written in the form

$$D_n n_1 - \beta_o |V_p| p_1 - CE_1 = 0 \quad , \tag{34}$$

$$D_p p_1 - \alpha_o |V_n| n_1 - CE_1 = 0 \quad , \tag{35}$$

where

$$D_n \equiv \left(j\omega - |V_n| \frac{\partial}{\partial x} - \alpha_o |V_n|\right) \quad , \tag{36}$$

$$D_p \equiv \left(j\omega + |V_p| \frac{\partial}{\partial x} - \beta_o |V_p|\right) \quad , \tag{37}$$

$$C \equiv \alpha'_o |V_n| N_o + \beta'_o |V_p| P_o \quad . \tag{38}$$

From Eqs. (34) and (35) it can be shown that

$$(D_n D_p - \alpha_o \beta_o |V_n| |V_p|)\, n_1 = C\left(j\omega + |V_p| \frac{\partial}{\partial x}\right) E_1 \quad , \tag{39}$$

$$(D_n D_p - \alpha_o \beta_o |V_n| |V_p|)\, p_1 = C\left(j\omega - |V_n| \frac{\partial}{\partial x}\right) E_1 \quad . \tag{40}$$

Thus, if Eq. (17), which we repeat here,

$$\frac{\partial E_1}{\partial x} = \frac{q}{\epsilon} (p_1 - n_1) \quad , \tag{17}$$

is operated on by $D_n D_p - \alpha_o \beta_o |V_n| |V_p|$, the result is

$$\frac{\partial}{\partial x}\left[(D_n D_p - \alpha_o \beta_o |V_n| |V_p|) + \frac{qC(|V_n| + |V_p|)}{\epsilon}\right] E_1 = 0 \quad , \tag{41}$$

which expands into

$$\frac{\partial}{\partial x} \Bigg\{ (|V_n| |V_p|) \frac{\partial^2}{\partial x^2} + [j\omega(|V_n| - |V_p|) + |V_n| |V_p| (\alpha_o - \beta_o)] \frac{\partial}{\partial x}$$

$$+ [\omega^2 + j\omega(\alpha_o |V_n| + \beta_o |V_p|)] - \frac{q}{\epsilon} (\alpha'_o |V_n| N_o + \beta'_o |V_p| P_o) \Bigg\} E_1 = 0 \quad . \tag{42}$$

When $\alpha = \beta$ and $|V_n| = |V_p|$, Eq. (42) reduces to the simpler form

$$\frac{\partial}{\partial x} \left(\frac{\partial^2}{\partial x^2} + K_m^2\right) E_1 = 0 \quad , \tag{43}$$

where $K_m^2 = (\omega/|V_n|)^2 + 2j\alpha_o(\omega/|V_n|) - 2\alpha'_o J_o/\epsilon |V_n|$.

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14. KEY WORDS

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